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Dimension on Discrete Spaces

Alexander V. Evako[1]

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In this paper we develop some combinatorial models for continuous spaces. We study the approximations of continuous spaces by graphs, molecular spaces, and coordinate matrices. We define the dimension on a discrete space by means of axioms based on an obvious geometrical background. This work presents some discrete models of n-dimensional Euclidean spaces, n-dimensional spheres, a torus, and a projective plane. It explains how to construct new discrete spaces and describes in this connection several three-dimensional closed surfaces with some topological singularities. It also analyzes the topology of $(3+1)$-spacetime. We are also discussing the question by R. Sorkin about how to derive the system of simplicial complexes from a system of open coverings of a topological space.

1. INTRODUCTION

A number of workers have been unhappy about applications of the continuum picture of space and spacetime. They believe that the breakdown of the functional integral at the Planck length shows not merely the failure of the classical field equations, but also that a differential manifold upon which they are built should be replaced by some finite theory. This was certainly one of the motivations behind Penrose's (1971) invention of spin networks and recent works by Finkelstein (1989) on a novel spacetime microstructure. Isham *et al.* (1990) introduce a quantum theory on the set of all topologies on a given set, and show that for a finite basic set almost all metrics can be obtained by embedding this set into a vector space and then varying the norm of this space.

Another approach to a combinatorial model of space and spacetime is studied by Sorkin (1991). He replaces general topological spaces by finite ones and describes how to associate a finite space with any locally finite

[1]Department of Physics, Syracuse University, Syracuse, New York 13244-3901.

covering of a topological space. He also presents some examples of posets derived from simple spaces. Some features of this approach are discussed by Balachandra *et al.* (1993).

Another approach is Regge calculus (Regge, 1961). Many suggestions for formulating various Regge calculus versions have been made in order to face a number of problems. Regge calculus describes general relativity spacetime by using a simplicial complex. Its fundamental variables are a set of edge lengths and an incidence matrix that describes how they are connected. One approach supposes that the connectivity of a simplicial complex is fixed, but the lengths of edges can be varied. Another approach fixes the edge length and varies the connectivity of a simplicial complex in order to change the metric of a spacetime. The Regge calculus, however, supposes that there is a continuous underlying spacetime and does not account naturally for the appearance of a minimal length in effective theories.

At the same time in mathematics there exist several recently developed approaches to discrete spaces in the frame of digital topology which can be useful in physics. Digital topology is the study of topological properties of image arrays. It provides the theoretical foundations for image processing operations such as image thinning, border following, and object counting. Kong and Rosenfeld (1989) review the fundamental concepts of digital topology, survey the major theoretical results in this field, and give a bibliography of almost 140 references.

Traditionally a discrete or digital space is considered as a graph whose edges between vertices define the nearness and connectivity in the neighborhood of any vertex.

This approach was used by Rosenfeld (1970, 1979), who proved the first version of the Jordan curve theorem by using a graph-theoretic model of a digital plane. However, this model does not utilize a topological basis and requires different nearness for the curve and its background.

An alternative topological approach to digital topology uses the notion of a connected topology on a totally ordered set Z of integers (Halimski, 1977; Khalimsky *et al.*, 1990; Kong *et al.*, 1991; Kopperman *et al.*, 1991). The digital plane $Z \times Z$ or the three-dimensional digital space $Z \times Z \times Z$ are the topological products of two or three such spaces, respectively. Using this construction, the Jordan curve theorem in two and three dimensions was proven. Another approach to finite topology is offered by Kovalevsky (1989). He builds the digital space as a structure consisting of elements of different dimensions by using such a well known element in topology as a cellular complex.

Our approach to discrete spaces is based on three combinatorial tools: (Ivaschenko, 1984, 1988, 1994, 1993*a,b*; Ivashchenko and Yeh, 1994). The material presented below begins with a short description of some geometrical background for the definition of the dimension on a graph. Then we show the connection between a graph, a molecular space, and a coordinate matrix. We define the dimension on discrete spaces which is based on some geometrical ground. We analyze the dimensions of different models of two-, three-, and n-dimensional discrete spaces. We present some examples of three-dimensional discrete closed spaces with strange topological features which do not have direct continuous analogies. Then we prove some theorems showing how to construct closed three-dimensional spaces with nonstandard topology. Finally we discuss the topological structure of a $(3+1)$ space-time.

2. GEOMETRICAL BACKGROUND FOR THE DEFINITION OF THE DIMENSION ON A GRAPH

We construct now a graph with certain properties which can be thought of as a convenient tool for describing the ideas of nearness and continuity by combinatorial methods. This will be done first by picking out in elementary geometry those properties of nearness which seem to be fundamental and taking them as axioms. To get a glimpse of the intuitive geometrical ground of the dimension consider the following example. Let E^n be n-dimensional Euclidean space and p a point in it. The neighborhood of p is commonly defined to be any set U such that U contains an open solid disk D_1^n of center p. The boundary of this disk is the sphere S_1^{n-1}.

The definition of neighborhood is formulated in this way so as to be as free as possible from the ideas of size and shape, concepts that play no part in topology.

Using this definition of a neighborhood of a p in Euclidean space, it is easy to see that the family of sets U satisfy the usual topological axioms.

1. p belongs to any neighborhood of p.
2. If U is a neighborhood of p and $U \subset V$, then V is a neighborhood of p.
3. If U and V are neighborhoods of p, so is $U \cap V$.
4. If U is a neighborhood of p, then there is a neighborhood V of p such that $V \subset U$ and V is a neighborhood of each of its points.

Taking these properties as axioms in an abstract formulation, we can define a topological space E as a set E with a family of subsets of E satisfying the four properties listed above. We can also define a subset W of E open if for each point p in W, W is a neighborhood of p.

Note that the disk D_1^n plays a crucial role in this definition.

In the continuous case the sphere S_1^{n-1} contains in itself an infinite sequence of disks D_i^n and spheres S_i^{n-1} of center p:

$$D_1^n \supset D_2^n \supset \cdots \supset D_i^n \supset \cdots$$

$$S_1^{n-1} \supset S_2^{n-1} \supset \cdots \supset S_i^{n-1} \supset \cdots$$

However, the situation is different in the discrete case, where the sequences of disks and spheres cannot be infinite and axiom 4 is not realized. Therefore we have a finite series of the form

$$D_1^n \supset D_2^n \supset \cdots \supset D_t^n$$

$$S_1^{n-1} \supset S_2^{n-1} \supset \cdots \supset S_t^{n-1}$$

The smallest disk D_t^n and the smallest sphere S_t^{n-1} cannot be reduced in the sense that they do not contain disks and spheres others than themselves (Fig. 1).

The topological meaning of this construction for a graph reveals that the vertex p is considered as n-dimensional if its minimal punctured neighborhood is the sphere S^{n-1}.

The point p and the nearest sphere S^{n-1} together form the smallest disk D^n of center p. Point p of a discrete space G is considered as one-dimensional if its nearest neighborhood is a zero-dimensional sphere S^0. It is well known that S^0 is a set of two disconnected points. In the other words S^0 is a disjoined graph of two points. In a one-dimensional discrete sphere S^1 all points are one-dimensional. Obviously the minimal number of points required for S^1 is four. For a two-dimensional discrete sphere S^2 all its points are two-dimensional. This means that the nearest neighborhood of any point of S^2 should be S^1 and so on.

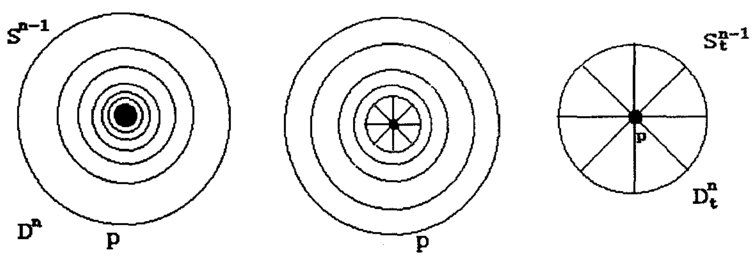

Fig. 1. Difference between an infinite and a finite number of enclosed disks D^n in continuous and discrete spaces, respectively.

3. A MOLECULAR SPACE AND A COORDINATE MATRIX OF A GRAPH

In order to make this paper self-contained we shall summarize the necessary results from our previous papers. Let E^∞ be infinite-dimensional Euclidean space. Take the coordinates of a point $x, x \in E^\infty$, as a sequence of real numbers

$$x = (x_1, x_2, \ldots, x_n, \ldots) = [x_i], \qquad i \in N$$

We define a unit cube $K \in E^\infty$ in the following way: each $x, x \in K$, has coordinates x_i satisfying conditions presented in Ivashchenko (1984, 1988, 1994, 1993*b*):

$$n_i \leq x_i \leq n_i + 1, \qquad i \in N, \quad n_i \text{ integer}$$

Therefore, K is an infinite-dimensional cube with unit edges. In Ivashchenko (1984, 1988, 1994, 1993*b*) K is called a brick (*kirpich*). We will use this name in the present paper.

The position of K in E^∞ is determined by the left vertex coordinates. For the given brick we have

$$K = (n_1, n_2, \ldots, n_n, \ldots) = [n_i], \qquad i \in N$$

Two bricks are called adjacent if they have common points. The distance $d(K_1, K_2)$ between bricks $K_1 = [n_i]$ and $K_2 = [m_i]$ is defined by using sup norm

$$d(K_1, K_2) = \max|n_i - m_i|, \qquad i \in N$$

Obviously, two bricks are adjacent if their appropriate coordinates differ by not more then 1, or the distance between them equals 1. Any set of bricks in E^∞ is called a molecular space and is denoted by M. Clearly, any molecular space can be represented by its intersection graph $G(M)$. It was shown in Ivashchenko (1988, 1993*b*) that any graph G can be represented by a molecular space $M(G)$ such that $G = G(M(G))$. Clearly, more than one $M(G)$ can be built for the graph G. There exists an isomorphism between any two $M(G)$. Let M be a molecular space with a set of bricks

$$V = (K_1, K_2, \ldots, K_n), \qquad K_1 = [k_{1i}], \qquad K_2 = [k_{2i}], \ldots, \qquad K_n = [k_{ni}]$$

The matrix $[k_{pi}]$ is called the coordinate matrix of the molecular space M and its intersection graph $G(M)$ and is denoted $A(M)$ or $A(G(M))$. This matrix has n rows and infinite columns.

In fact we shall always use a finite-dimensional Euclidean space. The intuitive background for using the infinite-dimensional unit cube is the attempt to create some universal element not depending on the dimension

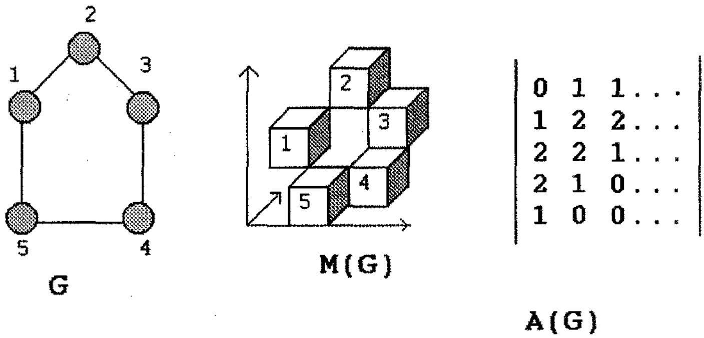

Fig. 2. Graph G, its molecular space $M(G)$, and its coordinate matrix $A(G)$.

and suitable for describing elements of different dimensions: zero-dimensional point, one-dimensional lines, two-dimensional surfaces, and so on.

Let S be a surface in E^n. The molecular space $M(S)$ of S is the set of bricks intersecting S.

Figure 2 shows the graph G, its molecular space M, and its coordinate matrix A.

4. THE DIMENSION AND THE METRIC OF A DISCRETE SPACE

Our objective now is to define the dimension on graphs. Later we will use the names discrete space and point for a graph and its vertex when we want to emphasize the notion of the dimension on it.

Since in this paper we only use induced subgraphs, we shall use the word subgraph for an induced subgraph. We shall also use some symbols, notations, and names introduced in our previous work.

Definitions. Let G, G_1, and v be a graph, a subgraph of G, and a point of G, respectively.

- The subgraph $B(G_1)$ containing G_1 is called the ball of G_1 if any point of $B(G_1)$ is adjacent to at least one point of G_1.
- The subgraph of $B(G_1)$ without points of G_1 is called the rim of G_1 and is denoted by $O(G_1)$.

Obviously $B(G_1) - G_1 = O(G_1)$.

- If G_1 is a point v, then $B(v)$ and $O(v)$ are called the ball and the rim of v, respectively.
- The subgraph $B(v_1, v_2, \ldots, v_n) = B(v_1) \cap B(v_2) \cap \cdots B(v_n)$ is called the joint ball of the points $v_1, v_2, \ldots, v_n$.

• The subgraph $O(v_1, v_2, \ldots, v_n) = O(v_1) \cap O(v_2) \cap \cdots O(v_n)$ is called the joint rim of the points $v_1, v_2, \ldots, v_n$.

Let G and G_1 be a graph and its subgraph with points $(v_1, v_2, \ldots, v_n)$ and $(v_1, v_2, \ldots, v_p)$, respectively. It is clear that

$$O(G_1) = O(v_1) \cup O(v_2) \cup \cdots \cup O(v_p) - G_1$$

$$B(G_1) = B(v_1) \cup B(v_2) \cup \cdots \cup B(v_p)$$

• A graph $K_n(v_1, v_2, \ldots, v_n)$ of n points is called completely connected or complete if any of its two points are adjacent.

• A graph $H_n(v_1, v_2, \ldots, v_n)$ of n points is called completely disconnected if any two of its points are disjoined.

• The join $G_1 * G_2$ of two graphs G_1 and G_2 is the graph which consists of two graphs G_1 and G_2 and all edges joining points of G_1 with points of G_2. To this end we begin by defining the dimension on a graph in the following way.

Definition 1. Zero-dimensional normal space S^0 is the graph which consists of two nonadjacent points.

Definition 2. A point v of a graph G is called a normal n-dimensional point if its rim $O(v)$ is a normal $(n-1)$-dimensional space.

Definition 3. For any integer n, $n \geq 1$, define a normal n-dimensional space to be a connected graph in which every point is n-dimensional normal.

According to these definitions, one-dimensional normal space is any circle C_n, $n \geq 4$.

Further, $p(G)$ will be used for the dimension of a graph G.

Figure 3 represents normal zero-, one-, and two-dimensional spheres and one-, two-, and three-dimensional disks.

Figure 4 shows normal two-dimensional discrete flat spaces and their molecular spaces. E_1^2 is the two-dimensional discrete space in Khalimsky topology (Halimsky, 1977; Khalimsky *et al.*, 1990; Kong and Rosenfeld, 1989; Kong *et al.*, 1991; Kopperman *et al.*, 1991).

The three-dimensional normal sphere is the graph S^3 depicted in Fig. 5. It can be verified without difficulties that the complete $(n+1)$ partite graph $K(2, 2, \ldots, 2)$ is the minimal graph describing S^n (Ivashchenko, 1984, 1988; Ivashchenko and Yeh, 1994). Therefore, the minimal number of elements necessary to describe S^n is $2n + 2$. Notice that the same number of points is used by Sorkin (1991) to describe S^n in the finitary topology approach.

A normal torus T^2 and a projective plane P^2 are presented in Fig. 5.

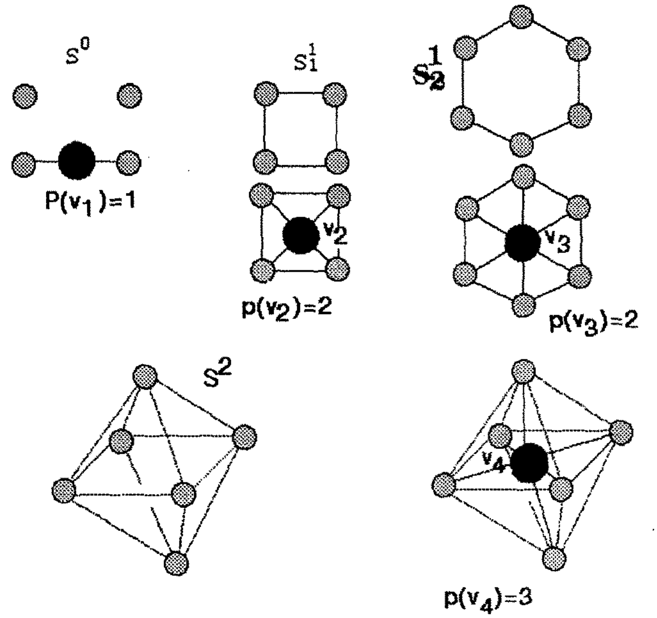

Fig. 3. Zero- (S^0), one- (S_1^1, S_2^1), and two- (S^2) dimensional normal discrete spheres, and one- (v_1), two- (v_2, v_3), and three- (v_4) dimensional points.

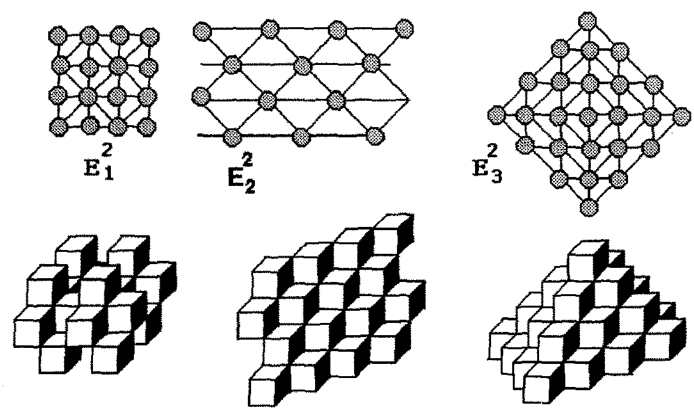

Fig. 4. Normal discrete two-dimensional planes and their molecular spaces. E_1^2 is the two-dimensional plane in Khalimsky topology.

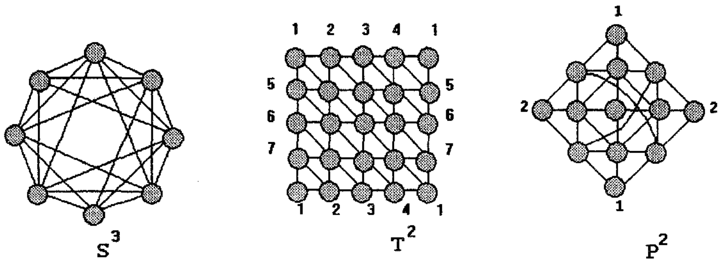

Fig. 5. A discrete normal three-dimensional sphere S^2, a two-dimensional torus T^2, and a two-dimensional projective plane P^2. The Euler characteristic and the homology groups of these graphs are consistent with the Euler characteristic and the homology groups of their continuous counterparts.

It can be checked directly that the Euler characteristic and the homology groups of all graphs depicted in Figs. 3–5 match the Euler characteristic and the homology groups of their continuous counterparts (Ivashchenko, 1994, 1993*a*).

In Ivashchenko (1984) normal n-dimensional space is called type Π_2. This separation into different types is caused by the fact that the normal molecular spaces and graphs of any type Π_n, $n \neq 1, 2$, have some unusual properties different from those of direct discrete models of continuous spaces in E^m.

Our objective now is to define a generalization of the dimension which includes the above definition. It is natural to consider a point n as zero-dimensional if its neighborhood does not contain any normal space.

Definition 4. A point v of a graph G is called zero-dimensional, $p(v) = 0$, if $O(v)$ does not contain the normal zero-dimensional sphere S^0.

Definition 5. A connected graph G is called zero-dimensional, $p(G) = 0$, if each of its points is zero-dimensional.

By this definition, in a zero-dimensional connected graph any two points are adjacent. Therefore, this graph is a complete graph on any number of vertices. A disconnected zero-dimensional graph is considered as a zero-dimensional sphere S^0 if it has exactly two components. It is clear that S^0 contains a normal zero-dimensional sphere as its subgraph. We will extend this analogy to higher dimensions.

Definition 6. A graph G is called closed n-dimensional if:

1. For any point v, $p(v) \leq n$.
2. G is homotopic to some normal n-dimensional space.

Definition 7. A point v is called n-dimensional, $p(v) = n$ if:
1. $O(v)$ contains a closed $(n-1)$-dimensional space.
2. $O(v)$ does not contain any closed n- or higher-dimensional space.

Definition 8. A graph G is called n-dimensional, $p(G) = n$, if:
1. G contains at least one n-dimensional point.
2. For any point v, $p(v) \leq n$.

In Definition 6 we use the homotopy of graphs. Two graphs are called homotopic if each of them can be turned into the other by contractible transformations which consist of contractible gluing and deleting of vertices and edges of a graph. It was shown (Ivashchenko, 1994, 1993*a*; Ivashchenko and Yeh, 1994) that these transformations do not change the Euler characteristic and the homology groups of graphs.

Let us look at some examples of n-dimensional (not normal) discrete spaces and their molecular spaces.

Spheres S^0, S^1, their molecular spaces, and the molecular space $M(S^2)$ of sphere S^2 are drawn in Fig. 6. $M(S^2)$ is a hollow space, it does not contain the central unit cube. These spheres are not normal but satisfy Definitions 6 and 7. Any sphere S^n depicted in Fig. 6 has the same Euler characteristic and homology groups as continuous S^n and can be transformed to the sphere S^n drawn in Fig. 3 by contractible transformations. Flat one-, two-, and three-dimensional spaces and their molecular spaces are shown in Fig. 7. It is easy to construct three- and higher-dimensional spaces, but it is difficult to draw them. For a flat three-dimensional space

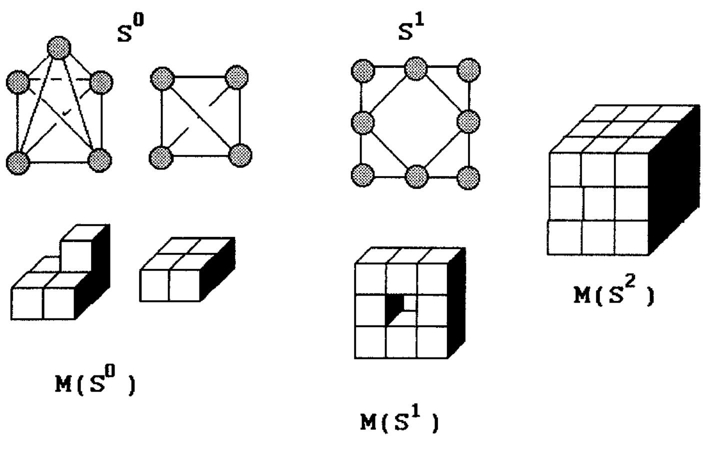

Fig. 6. Zero- and one-dimensional nonnormal spheres S^0 and S^1 and their molecular spaces $M(S^0)$ and $M(S^1)$. $M(S^2)$ is a molecular space of the two-dimensional nonnormal sphere S^2. It does not contain the central unit cube.

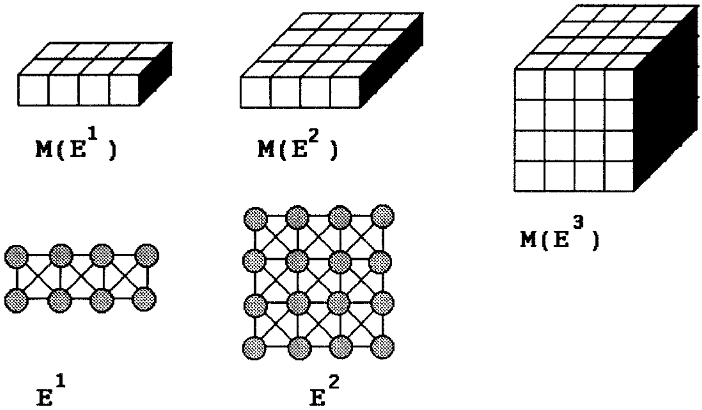

Fig. 7. Nonnormal discrete one- and two-dimensional flat spaces E^1 and E^2 and their molecular spaces $M(E^1)$ and $M(E^2)$. $M(E^3)$ is a molecular space of a nonnormal discrete three-dimensional flat space E^3.

only the molecular space is shown. However, an n-dimensional space can be easily described by its coordinate matrix of the form

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pn} \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

where $x_{ik} = 0, \ \pm 1, \ \pm 2, \ldots ; \ i = 1, 2, 3, \ldots ; \ k = 1, 2, \ldots, n$.

The standard definition of the distance on a graph can be applied to a discrete space.

Definition 9. The distance $d(v_1, v_2)$ between two points v_1 and v_2 in a discrete space G is the length of the shortest path joining them if any; otherwise $d(v_1, v_2) = \infty$.

Obviously the distance is a metric. The Planck length can be thought of as the length of an edge of the graph.

5. MATHEMATICAL OBSERVATIONS

Before proceeding to the main result of this paper, let us pause to describe some mathematical observations relating to this approach. The following surprising facts were revealed.

• Suppose that S^1 is a circle of radius R. Let A be a cover of S^1 by arcs whose length is small enough compared with R. Denote $G(A)$ the intersection graph of this cover. This graph is called the circular arc graph. It appears that:

1. The dimension of $G(A)$ is equal to one, $p(G(A)) = \dim(S^1) = 1$.
2. $G(A)$ has the same Euler characteristic and homology groups as S^1.
3. $G(A)$ can be reduced to the cycle graph C_4 by contractible transformations (Ivashchenko, 1994, 1993*a*; Ivashchenko and Yeh, 1994) (S_1^1 in Fig. 3).

• Suppose we have some two-dimensional closed surface, for example, a sphere S^2 of radius R. Consider any tiling A of S^2 by elements $(a_1, a_2, \ldots, a_n)$ whose size is small enough relative to the radius R. Construct the intersection graph $G(A)(v_1, v_2, \ldots, v_n)$ in the following way: Two vertices v_1 and v_2 are adjacent iff elements a_1 and a_2 have at least one common point. In most cases it turns out that:

1. The dimension of $G(A)$ is equal to two, $p(G(A)) = \dim(S^2) = 2$.
2. $G(A)$ has the same Euler characteristic and homology groups as S^2.
3. $G(A)$ can be reduced by contractible transformations into the minimal two-dimensional sphere on six vertices (Ivashchenko, 1994, 1993*a*; Ivashchenko and Yeh, 1994) (S^2 in Fig. 3).

• Suppose that P^k is a surface in E^n, $n = 2, 3$ (for spheres, n can be any number). Divide E^n into a set of cubes with the scale l_1 of the cube edge and call the molecular space $M_1(P^k)$ of P^k the family of cubes intersecting P^k. Denote $G_1(P^k)$ the intersection graph of $M_1(P^k)$. Change the scale of the cube edge from l_1 to l_2 and obtain $M_2(P^k)$ and $G_2(P^k)$ by using the same structure. It is revealed that in most cases:

1. $p(G_1(P^k)) = p(G_2(P^k)) = \dim(P^k)$.
2. $G_1(P^k)$ and $G_2(P^k)$ have the same Euler characteristic and the homology groups as P^k.
3. $G_1(P^k)$ and $G_2(P^k)$ can be transformed from one to the other with four kinds of transformations if the divisions are small enough.

These facts allow us to assume that the graph and the molecular space contain topological and perhaps geometrical characteristics of the surface P^k. Otherwise, the molecular space M and the graph G are the discrete counterparts of a continuous space P^k.

6. SINGULAR SPACES

This section describes a method of obtaining new spaces from given ones. We will see that there exist n-dimensional normal spaces with some peculiar properties. These spaces give rise to new discrete structures that have different topologies in different points.

Theorem 1. Let $G(p_1, p_2, \ldots, p_r)$ and $H_2(v_1, v_2)$ be an n-dimensional normal space and the completely disconnected space on two points, respectively. Then $H_2(v_1, v_2) * G(p_1, p_2, \ldots, p_r)$ is an $(n+1)$-dimensional normal space.

Proof. The proof is by induction.

(i) For $n = 0, 1$ the theorem is verified directly.

(ii) Assume that the theorem is valid for any n, $n \le k$. Let $G(p_1, p_2, \ldots, p_r)$ be a normal $(k+1)$-dimensional discrete space. Consider

$$W = H_2(v_1, v_2) * G(p_1, p_2, \ldots, p_r)$$

It is necessary to show that W is a $(k+2)$-dimensional discrete normal space. Take any point p_i. With respect to the definition of a normal space, $O(p_i)$ in G, denoted $O(p_i) \mid G$, is a k-dimensional normal space. Therefore, according to the assumption, $H_2 * O(p_i)$ is a $(k+1)$-dimensional normal space. Hence any point p_i in W has a rim which is a $(k+1)$-dimensional normal space.

The rims of points v_1 and v_2 in W are the $(k+1)$-dimensional normal space G by construction. We have

$$O(p_i) \mid W = H_2 * (O(p_i) \mid G), \quad i = 1, 2, \ldots, n, \qquad O(v_k) \mid W = G, \quad k = 1, 2$$

Therefore, the rim of any point of W is a $(k+1)$-dimensional normal space and, by the definition, W is a normal $(k+2)$-dimensional space. This completes the proof. ■

We are now in a position to describe n-dimensional normal spaces with peculiar properties.

• First construct a space without singularities. Let G be an n-dimensional sphere S^n. This means that the rim of any point of S^n is a normal sphere S^{n-1}, and S^n can be turned into the minimal S^n on $2n+2$ points by contractible transformations (Ivashchenko, 1984, 1993*a*; Ivashchenko and Yeh, 1994).

Consider $W = H_2(v_1, v_2) * S^n$. If $p \in S^n$, then $O(p) \mid W = H_2(v_1, v_2) * S_p^{n-1} = S_p^n$. For points v_1 and v_2 the rim is S^n itself. Therefore, the rim of any point of W is a sphere S^n, and W is a normal $(n+1)$-dimensional space. It is easy to show that W can be reduced to the minimal $(n+1)$-sphere S^{n+1} by contractible transformations and therefore $W = S^{n+1}$:

$$W = H_2 * S^n = S^{n+1}$$

$$O(p) \mid W = H_2 * S_p^{n-1} = S_p^n, \quad p \in S^n; \qquad O(v_1) \mid W = O(v_2) \mid W = S^n$$

• Suppose that G is a discrete two-dimensional torus T^2 depicted in Fig. 5. For any point p of T^2 $O(p) = S_p^1$. Therefore, in $W = H_2(v_1, v_2) * T^2$

the rim of any point p is a two-dimensional sphere S_p^2, $O(p) \mid W = S_p^2$. However, for points v_1 and v_2 their rims are the torus T^2 itself, $O(v_i) = T^2$, $i = 1, 2$. Notice that the dimension of T^2 is equal to 2. Hence W is a normal three-dimensional space in which the rims of points have a different topology. For points v_1 and v_2 the space has torus neighborhood T^2; in all other points the neighborhood is spherical, S^2. We have

$$W = H_2 * T^2$$

$$O(p) \mid W = H_2 * S_p^1 = S_p^2, \quad p \in T^2; \qquad O(v_1) \mid W = O(v_2) \mid W = T^2$$

• Another peculiar three-dimensional space appears when we choose the projective plane P^2 (Fig. 5) as a basic space G.

In three-dimensional normal space $W = H_2(v_1, v_2) * P^2$ the neighborhoods of v_1 and v_2 are the projective plane P^2; the neighborhoods of all other points are usual spheres S^2. We have

$$W = H_2 * P^2$$

$$O(p) \mid W = H_2 * S_p^1 = S_p^2, \quad p \in P^2; \qquad O(v_1) \mid W = O(v_2) \mid W = P^2$$

• In general we can create a number of three-dimensional normal spaces with two singularities by taking discrete models of closed two-dimensional oriented or nonoriented surfaces as a basic space.

7. THE DIMENSIONAL LOCAL STRUCTURE OF A PHYSICAL DISCRETE (3 + 1) SPACE-TIME

Now we are ready to discuss some general features of the physical (3 + 1) space-time. We will restrict our consideration to local properties of a point v.

Theorem 2. (3 + 1) space-time is four-dimensional nonnormal.

Proof. We have to prove that in (3 + 1) space-time the rim of any point is a closed three-dimensional nonnormal discrete space.

Suppose that a physical object is at a point v of a three-dimensional discrete space $R(t)$ at a given moment t and at either the same or the nearest point v_1 at the next moment $t + Dt$ (Fig. 8a). In (3 + 1) space-time (R, T) we have two three-dimensional spaces $R(t)$ and $R(t + Dt)$ corresponding to the different moments. Obviously these spaces are joined together in the following way. Point v on $R(t)$ should be connected with the ball $B(v)$ on $R(t + Dt)$ (Fig. 8b). Therefore, in the (3 + 1) space-time (R, T) (Fig. 8c) the rim $O(v) \mid (R, T)$ of point v is as shown in Fig. 8d.

(i) If the rim $O(v) \mid R(t)$ of v in $R(t)$ is a nonnormal closed two-dimensional space, then, for the same reasons as in Theorem 1, $O(v)$ in

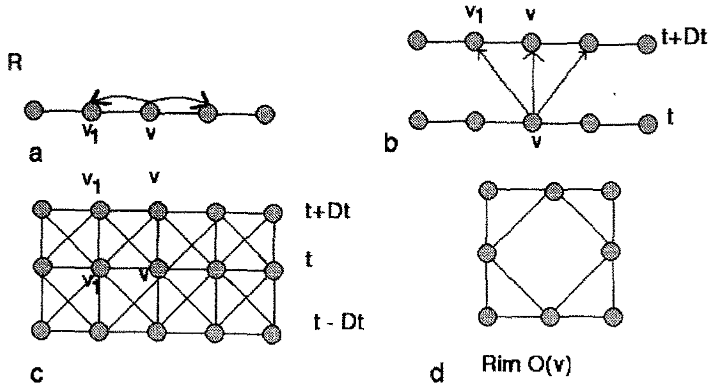

Fig. 8. Theorem 2 for $(1+1)$ space-time. $(1+1)$ space-time is not normal because $O(v)$ is not a normal one-dimensional sphere.

(R, T) is a nonnormal closed three-dimensional space, and (R, T) is a nonnormal four-dimensional space.

(ii) Suppose that $R(t)$ is a normal three-dimensional space. Then $O(v)$ in $R(t)$ is a normal two-dimensional discrete space. Obviously $O(v)$ in (R, T) contains the normal three-dimensional space $H(u_1, u_2) * O(v) \mid R(t)$, where u_1 and u_2 are v in $R(t+Dt)$ and $R(t-Dt)$. By Theorem 1 it is a normal three-dimensional space. Take v_1 in $R(t+Dt)$, $v_1 \in R(t+Dt)$, $v_1 \in O(v) \mid (R, T)$. It is easy to see that v in $R(t+Dt)$ is adjacent to all points of the rim of this v_1 in $O(v) \mid (R, T)$. Hence $O(p) \mid (R, T)$ is a nonnormal closed three-dimensional space which can be reduced to normal $H(u_1, u_2) * O(v) \mid R(t)$ by contractible transformations. Thus (R, T) is a nonnormal four-dimensional space-time. ■

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